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The force on an axisymmetric body in linearized, time-dependent motion: a new memory term

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In contrast to the steady Stokes equations for creeping motion, the time-dependent linearized Navier–Stokes equations have only been solved for very restricted geometries, the solution for the sphere being the sole solution for an isolated finite body. In the present paper, the linearized Navier–Stokes equations are further explored and a simple expression is derived which relates the force on an arbitrary axisymmetric body in oscillatory motion to the solution for the stream function in the far field. This result is applied to the case of a slightly eccentric spheroid and it is shown that the total hydrodynamic force contains four terms, three of which correspond to the classical solutions for the Stokes drag, added mass and Basset force on the perturbed sphere; the fourth term is only present when the body is non-spherical. In contrast to the three classical forces, the new term is not a simple power of the dimensionless frequency parameter $\mathrm{i}L^2\omega/\nu$, in which L is a length-scale, ω is the frequency of oscillation and ν is the kinematic viscosity of the fluid. A Laplace superposition is then used to find the force on the spheroid in an arbitrary axisymmetric motion with velocity $U(t)$. The new memory term decays faster than the Basset force at large times and is bounded at short times.

1. Introduction

The linearized Navier–Stokes equations were first solved for the oscillatory motions of a sphere along a diameter, a cylinder along a diameter, and a flat wall in its own plane by Stokes (1851). In particular, he found the hydrodynamic force on the oscillating sphere to be

$$F = \mathrm{Re}\{-6\pi\mu Ua(1+\lambda+\tfrac{1}{9}\lambda^2)\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{1}$$

in which μ is the fluid viscosity, U and a are the peak velocity and radius of the sphere and λ is a dimensionless parameter given by

$$\lambda^2 = -\mathrm{i}\frac{a^2\omega}{\nu} \quad (\mathrm{Re}\{\lambda\} > 0). \tag{2}$$

The frequency of oscillation is ω and ν is the kinematic viscosity of the fluid. Basset (1888) integrated (1) to obtain the force on a sphere in an arbitrary time-dependent motion with velocity $U(t)$:

$$F = -6\pi\mu aU + 6\pi\mu a^2\frac{1}{(\pi\nu)^{\frac{1}{2}}}\int_0^t \frac{\mathrm{d}U}{\mathrm{d}\tau}(t-\tau)^{-\frac{1}{2}}\,\mathrm{d}\tau - \tfrac{2}{3}\pi\rho a^3\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{3}$$

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Stokes formula (1) contains three terms and is a simple quadratic function of $\omega^{\frac{1}{2}}$. The first term is in phase with U and is equal to the so-called steady Stokes drag. The third term, which is proportional to ω, is $\frac{1}{2}\pi$ out of phase with U and is the non-dissipative added-mass force. The middle term, which has a phase lag of $\frac{1}{4}\pi$, is proportional to $\omega^{\frac{1}{2}}$ and is the so-called Basset force. This term describes the growth of the time-dependent boundary layer at the body surface and the displacement effect of the viscous layer on the inviscid pressure field.

It is interesting to note that if the force on the sphere is calculated as an asymptotic series for small ω starting with the steady Stokes equations and including small inertial effects, the series terminates after three terms and yields equation (1). Similarly, if an asymptotic series is developed for large ω, starting with potential flow and including boundary-layer effects (cf. Batchelor 1967) equation (1) is also obtained wherein the damping force on the oscillating sphere can be identified as the part of (1) in phase with the velocity. Thus, for a sphere at least, the Basset force on an isolated body at all frequencies can be obtained by considering the first-order inertial correction to the Stokes drag, or the first-order boundary-layer correction to the virtual mass in small-amplitude motion.

Now for any body shape in axisymmetric flow, we can obtain the small-ω behaviour from the quasi-steady Stokes equations using the formula given by Payne & Pell (1960):

$$F = 8\pi \lim_{r\to\infty} \frac{r\psi}{\varpi^2}, \tag{4}$$

in which ψ is the stream function, r is the distance from the origin located at the centre of the body, and ϖ is the distance from the axis of symmetry. We can similarly obtain the large-ω behaviour from potential flow. The question which then arises is: 'Can we derive the middle term, i.e. the Basset force for an arbitrary body, as a first-order correction to the Stokes drag or the virtual mass?', or equivalently: 'Is the hydrodynamic force on an oscillating body of arbitrary shape a quadratic function of λ with different coefficients?' This possibility is suggested by the fact that the coefficients of both the steady term in phase with the velocity and the added mass term depend only on the geometry. If the answer to these questions is yes, then we have a very quick method for obtaining the hydrodynamic force on a variety of body shapes for which the Stokes drag and virtual mass are already known. Unfortunately, the answer is no; it is shown below that even for a nearly spherical body the force is not quadratic in λ and that the functional form of the Basset force and its phase depend on the body geometry.

First, the analogue of Payne & Pell's result (4) will be derived for an arbitrary axisymmetric body in oscillating flow. Then this result will be applied to a nearly spherical body to show how the eccentricity leads to a different functional form for the total Basset force on the body. This force will be shown to consist of two components, one of which is of the standard form of the Basset force and a second term which, when integrated over all frequencies, leads to a new force with a different memory function that has a non-singular behaviour at short times. For convenience, we choose the body to be an oblate spheroid of small eccentricity since exact solutions for a spheroid exist for both the steady Stokes drag and the added-mass force.

Since Stokes's celebrated solutions were published, very little new work has appeared for the unsteady linearized equation. All the problems described in the literature (Basset 1888; Batchelor 1967; Felderhof 1976*a*, *b*; Lin 1986; Mazur & Bedeaux 1974; Leichtberg *et al.* 1976) have dealt with one of the three basic geometries discussed by Stokes. It is believed that the solution presented herein is

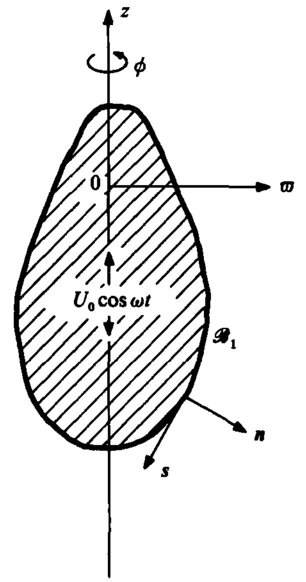

FIGURE 1. Axisymmetric body and cylindrical coordinate system (ϖ, ϕ, z).

the first for a different basic geometry which displays the non-quadratic behaviour of the λ-dependence for an oscillating body and a memory function for an arbitrary motion $U(t)$ which does not behave simply as $1/t^{\frac{1}{2}}$.

2. The hydrodynamic force for axisymmetric motion

We consider a simply connected smooth body oscillating axisymmetrically as shown in figure 1. The linearized equations of motion are

$$\nabla \cdot \boldsymbol{u} = 0, \quad \rho \frac{\partial \boldsymbol{u}}{\partial t} = \nabla \cdot \boldsymbol{\sigma}, \tag{5}$$

where $\boldsymbol{\sigma}$ is the stress tensor for the fluid and ρ is the fluid density. The boundary conditions are:

$$\boldsymbol{u} = \boldsymbol{U} = U_0 \boldsymbol{i}_z \cos \omega t \quad \text{on the body } \mathscr{B}_1, \tag{6}$$

$$\boldsymbol{u}, \boldsymbol{\sigma} \rightarrow 0 \quad \text{as } r \rightarrow \infty. \tag{7}$$

In (6) $\boldsymbol{i}_z$ is the unit vector along the z-axis and in (7) $r = (\varpi^2 + z^2)^{\frac{1}{2}}$. The linearization requires that the Reynolds number $Re = U_0 L/\nu$ should be small. Here L is a typical dimension of the body. The motion of the boundary need not be considered, since for an isolated body linearization enables us to add a uniform velocity and the velocity field is found in terms of the instantaneous coordinate system fixed in the body. This point is academic for periodic motion since the ratio of the maximum excursion to the body lengthscale is $O(Re/|\lambda^2|) \ll 1$ by assumption. However, for general motions, to be discussed later, the body may have a large net displacement, so the above argument is important.

We introduce a stream function in spherical coordinates, $\Psi(r, \theta, t)$ so that the velocity is given by

$$\boldsymbol{u} = -\nabla \wedge (\boldsymbol{i}_\phi \Psi/\varpi). \tag{8}$$

Then (5) can be replaced by

$$\nabla \cdot \boldsymbol{\sigma} = -\rho \nabla \wedge \left(\boldsymbol{i}_\phi \frac{1}{\varpi} \frac{\partial \Psi}{\partial t}\right). \tag{9}$$

Equation (9) is put in dimensionless form and the time-dependence is separated by introducing $\psi^*(r^*, \theta)$ and $\boldsymbol{\sigma}^*(r^*, \theta)$ defined by

$$\Psi(r,\theta,t) = \mathrm{Re}\{\psi^*(r^*,\theta)\,\mathrm{e}^{-\mathrm{i}\omega t}\}\,L^2U_0, \tag{10a}$$

$$\boldsymbol{\sigma}(r,\theta,t) = \mathrm{Re}\{\boldsymbol{\sigma}^*(\boldsymbol{r}^*,\theta)\,\mathrm{e}^{-\mathrm{i}\omega t}\}\,\mu U_0/L, \tag{10b}$$

in terms of dimensionless coordinates $\boldsymbol{r}^* = \boldsymbol{r}/L$. We now drop the asterisks to give

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = -\lambda^2\boldsymbol{\nabla} \wedge (\boldsymbol{i}_\phi \psi/\varpi). \tag{11}$$

Equation (11) is integrated over the domain $\mathscr{D}$ bounded by $\mathscr{B}_1$ and a large concentric sphere $\mathscr{B}_2$ of radius R. The volume integrals are changed to surface integrals using Stokes's theorem and we retain only the z-component of the vector equation:

$$\iint_{\mathscr{B}_1} \boldsymbol{i}_z\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}S + \iint_{\mathscr{B}_2} \boldsymbol{i}_z\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}S + \lambda^2 \iint_{\mathscr{B}_1} \boldsymbol{i}_z\cdot\boldsymbol{n} \wedge \boldsymbol{i}_\phi(\psi/\varpi)\,\mathrm{d}S + \lambda^2 \iint_{\mathscr{B}_2} \boldsymbol{i}_z\cdot\boldsymbol{n} \wedge \boldsymbol{i}_\phi(\psi/\varpi)\,\mathrm{d}S = 0. \tag{12}$$

Here $\boldsymbol{n}$ is the local outward unit normal to the domain $\mathscr{D}$. The first integral of (12) is simply the force exerted by the body on the fluid, $-F$, and the others may be evaluated individually as follows. For the second integral of (12) we follow the method of Happel & Brenner (1965) to put it in the form:

$$\iint_{\mathscr{B}_2} \boldsymbol{i}_z\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}S = \pi\int \varpi^3 \frac{\partial}{\partial n}\left(\frac{1}{\varpi^2}\mathrm{E}^2\psi\right)\mathrm{d}s - \pi\lambda^2\int \varpi\frac{\partial\psi}{\partial n}\mathrm{d}s, \tag{13}$$

in which s is the coordinate along the generating arc of $\mathscr{B}_2$, λ is defined by (2) and E^2 is the differential operator

$$\mathrm{E}^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial\zeta^2} \quad \text{with } \zeta = \cos\theta. \tag{14}$$

Now Basset (1888) has given the general solution of (5) subject to (7) in terms of spherical harmonics:

$$\psi = \psi^{\mathrm{P}} + \psi^{\mathrm{D}}, \tag{15}$$

with

$$\psi^{\mathrm{P}} = \sum_0^\infty A_n r^{-n+1}\mathscr{I}_n(\zeta), \tag{16a}$$

and

$$\psi^{\mathrm{D}} = \sum_0^\infty B_n R_n^-(r)\mathscr{I}_n(\zeta). \tag{16b}$$

The $\mathscr{I}_n(\zeta)$ are Gegenbauer functions of degree $-\frac{1}{2}$ of which the first few are:

$$\mathscr{I}_0(\zeta) = 1, \quad \mathscr{I}_1(\zeta) = -\zeta, \quad \mathscr{I}_2(\zeta) = \tfrac{1}{2}(1-\zeta^2). \tag{17}$$

The $R_n^-(r)$ are polynomials in $(1/r)$ with an exponential multiplier:

$$R_n^-(r) = r^n\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{n-1}\frac{1}{r}\mathrm{e}^{-\lambda r}. \tag{18}$$

The stream function satisfies the equation

$$\mathrm{E}^2(\mathrm{E}^2-\lambda^2)\psi = 0, \tag{19}$$

with

$$\mathrm{E}^2\psi^{\mathrm{P}} = 0 \quad \text{and} \quad (\mathrm{E}^2-\lambda^2)\psi^{\mathrm{D}} = 0, \tag{20}$$

so ψ^{P} and ψ^{D} may be thought of as potential and diffusive parts of the solution. If there are no sources, then A_0 is zero. The $R_n^-(r)$ are exponentially small at large r, so we may neglect them in (13) if R is large enough. Then (13) reduces to:

$$\iint_{\mathscr{B}_2} \boldsymbol{i}_z\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}S = -\pi\lambda^2\int \varpi\frac{\partial\psi^{\mathrm{P}}}{\partial r}\,\mathrm{d}s. \tag{21}$$

Finally, we use the property of the $\mathscr{I}_n$:

$$\int_{-1}^{1}\mathscr{I}_n(\zeta)\,\mathrm{d}\zeta = 2\delta_{n0}+\tfrac{2}{3}\delta_{n2}, \tag{22}$$

with δ_{nm} the Kronecker delta. We then have

$$\iint_{\mathscr{B}_2} \boldsymbol{i}_z\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\mathrm{d}S = \tfrac{2}{3}\pi\lambda^2A_2, \tag{23}$$

which is independent of R. The third integral in (12) is evaluated using the condition (6) on $\mathscr{B}_1$. The integral takes the value $\lambda^2\mathscr{V}$ with $\mathscr{V}L^3$ the volume of the body. The final integral in (12) is evaluated using a procedure that is similar to that just outlined for the second integral, and its value is determined to be $\tfrac{4}{3}\pi\lambda^2A_2$.

The above results are combined to give an expression for the force on the body:

$$F = \lambda^2(2\pi A_2+\mathscr{V}). \tag{24}$$

The term $\lambda^2\mathscr{V}$ is simply the inertial resistance of the displaced fluid. A_2 depends on λ and the geometry of the body and that term has no simple interpretation. From (16a) and (16b), we may express A_2 in the form:

$$A_2 = 2\lim_{r\to\infty}\frac{r^3\psi}{\varpi^2}, \tag{25}$$

to give:

$$F = \lambda^2\mathscr{V}+4\pi\lambda^2\lim_{r\to\infty}\frac{r^3\psi}{\varpi^2}. \tag{26}$$

This is analogous to the result (4) of Payne & Pell (1960) for steady Stokes flow. The limiting behaviour of (26) as $\lambda\to 0$ does not yield Payne & Pell's formula, because the limit is singular. Even for very low frequencies the far regions are effectively inviscid, a fact that is ignored in the steady Stokes equation. This situation is analogous to Oseen's result for the non-linear inertial correction to the Stokes drag in steady flow. In fact, if the limit $\lambda\to 0$ is taken before the limit $r\to\infty$, it can be shown that (26) does agree with (4).

3. The stream function for an oblate spheroid

An oblate spheroid with semi-axes $a = b(1+\epsilon)$, b is given by

$$\frac{\varpi^2}{a^2}+\frac{z^2}{b^2} = 1. \tag{27}$$

We choose lengthscale $L = b$ and the boundary is

$$r = r_0(\zeta) = 1+\epsilon 2\mathscr{I}_2(\zeta)+\epsilon^2 3(\mathscr{I}_2^2(\zeta)-\mathscr{I}_2(\zeta))+O(\epsilon^3). \tag{28}$$

We assume that $\epsilon \ll 1$ and it is convenient to move the boundary conditions from $r = r_0$ to $r = 1$ by expanding ψ and ψ_r in Taylor series about $r = 1$, retaining terms up to $O(\epsilon^3)$. If we now write ψ as a power series in ϵ,

$$\psi(r,\zeta;\epsilon) = \psi_0(r,\zeta)+\epsilon\psi_1(r,\zeta)+\epsilon^2\psi_2(r,\zeta)+O(\epsilon^3), \tag{29}$$

we have boundary-value problems for each ψ_i:

$$\mathrm{E}^2(\mathrm{E}^2-\lambda^2)\,\psi_i = 0 \quad (i = 0,1,2), \tag{30}$$

with

$$\psi_i/r^2 \to 0 \quad \text{as } r \to \infty \quad (i = 0,1,2), \tag{31}$$

and boundary conditions on $r = 1$:

$$\psi_0 = -\mathscr{I}_2(\zeta), \quad \psi_{0r} = 2\mathscr{I}_2(\zeta), \tag{32}$$

$$\psi_1 = 0, \quad \psi_{1r} = -2\mathscr{I}_2(\zeta)\,\psi_{0rr}, \tag{33}$$

$$\psi_2 = 4\mathscr{I}_2^3(\zeta) + 2\mathscr{I}_2^2(\zeta)\,\psi_{0rr}, \tag{34}$$

$$\psi_{2r} = -2\mathscr{I}_2\,\psi_{1rr} - [6\mathscr{I}_2^2 - 3\mathscr{I}_2]\,\psi_{0rr} - 2\mathscr{I}_2^3\,\psi_{0rrr} - 12\mathscr{I}_2^3 + 6\mathscr{I}_2^2. \tag{35}$$

We simplify the above by using the recurrence relation for $m \geqslant 2$:

$$\mathscr{I}_m\,\mathscr{I}_2 = -\frac{(m-2)(m-3)}{2(2m-1)(2m-3)}\mathscr{I}_{m-2} + \frac{m(m-1)}{(2m+1)(2m-3)}\mathscr{I}_m - \frac{(m+1)(m+2)}{2(2m-1)(2m+1)}\mathscr{I}_{m+2}. \tag{36}$$

Then suitable forms for the ψ_i are:

$$\psi_0(r,\zeta) = \mathscr{I}_2(\zeta) f_2^{(0)}(r), \tag{37}$$

$$\psi_1(r,\zeta) = \mathscr{I}_2(\zeta) f_2^{(1)}(r) + \mathscr{I}_4(\zeta) f_4^{(1)}(r), \tag{38}$$

$$\psi_2(r,\zeta) = \mathscr{I}_2(\zeta) f_2^{(2)}(r) + \mathscr{I}_4(\zeta) f_4^{(2)}(r) + \mathscr{I}_6(\zeta) f_6^{(2)}(r), \tag{39}$$

with conditions on $r = 1$:

$$f_2^{(0)} = -1, \quad f_{2r}^{(0)} = -2, \tag{40}$$

$$f_2^{(1)} = 0, \quad f_{2r}^{(1)} = -\tfrac{4}{5}(f_{2rr}^{(0)} + 2), \tag{41}$$

$$f_4^{(1)} = 0, \quad f_{4r}^{(1)} = \tfrac{4}{5}(f_{2rr}^{(0)} + 2), \tag{42}$$

$$f_2^{(2)} = \tfrac{12}{35}(f_{2rr}^{(0)} + 2), \quad f_{2r}^{(2)} = -\tfrac{4}{5} f_{2rr}^{(1)} + \tfrac{2}{25} f_{4rr}^{(1)} + \tfrac{6}{35}(f_{2rr}^{(0)} + 2) - \tfrac{12}{35} f_{2rrr}^{(0)}, \tag{43}$$

$f_4^{(2)}$ and $f_6^{(2)}$ make no contribution to the force up to $O(\epsilon^2)$ and we shall not need them. The radial dependence comes from (16a) and (16b) and we have:

$$f_2^{(i)} = A_2^{(i)}\frac{1}{r} + B_2^{(i)}\,\mathrm{e}^{-\lambda(r-1)}\left(1 + \frac{1}{\lambda r}\right) \quad (i = 0,1,2), \tag{44}$$

$$f_4^{(1)} = A_4^{(1)}\frac{1}{r^3} + B_4^{(1)}\,\mathrm{e}^{-\lambda(r-1)}\left(1 + \frac{6}{\lambda r} + \frac{15}{\lambda^2 r^2} + \frac{15}{\lambda^3 r^3}\right). \tag{45}$$

The constants $A_2^{(i)}$ are determined from (40)–(43) and are

$$A_2^{(0)} = -\left(1 + \frac{3}{\lambda} + \frac{3}{\lambda^2}\right), \tag{46}$$

$$A_2^{(1)} = -\frac{12}{5}\left(1 + \frac{1}{\lambda}\right)^2, \tag{47}$$

$$A_2^{(2)} = -\frac{6}{175}\left(53 + \frac{58}{\lambda} + \frac{1}{\lambda^2}\right) - \frac{24}{175}\left(\frac{1}{3 + 3\lambda + \lambda^2}\right). \tag{48}$$

And we can evaluate A_2 using

$$A_2 = \lim_{r\to\infty}\frac{r^3\psi}{\varpi^2} = A_2^{(0)} + \epsilon A_2^{(1)} + \epsilon^2 A_2^{(2)} + O(\epsilon^3). \tag{49}$$

4. Hydrodynamic force on a spheroid

The volume of the spheroid is

$$\mathscr{V} = \tfrac{4}{3}\pi(1+\epsilon)^2. \tag{50}$$

Using (49) and (50) in (24) we have for the force:

$$F = -6\pi\left[(1+\lambda+\tfrac{1}{9}\lambda^2)+\epsilon\tfrac{4}{5}(1+2\lambda+\tfrac{4}{9}\lambda^2)+\epsilon^2\tfrac{2}{175}\left(1+58\lambda+\tfrac{302}{9}\lambda^2+\frac{4\lambda^2}{3+3\lambda+\lambda^2}\right)+O(\epsilon^3)\right] \tag{51}$$

$$= -6\pi\left[(1+\tfrac{4}{5}\epsilon+\tfrac{2}{175}\epsilon^2)+\lambda(1+\tfrac{8}{5}\epsilon+\tfrac{116}{175}\epsilon^2)+\tfrac{1}{9}\lambda^2(1+\tfrac{16}{5}\epsilon+\tfrac{604}{175}\epsilon^2)\right.$$
$$\left.+\frac{8}{175}\left(\frac{\lambda^2\epsilon^2}{3+3\lambda+\lambda^2}\right)+O(\epsilon^3)\right]. \tag{52}$$

Equation (51) is the result for a sphere with the $O(\epsilon)$ and $O(\epsilon^2)$ corrections for the slightly oblate spheroid. It also holds for a prolate spheroid with $\epsilon < 0$. (Note that then a is the semi-minor axis.)

For small frequencies, $|\lambda| \ll 1$, we recover the Stokes resistance for a slightly oblate spheroid which is the first term in (52). This can be demonstrated using Happel & Brenner's (1965) exact expression for the force on an oblate spheroid:

$$F = -8\pi c[x_0-(x_0^2-1)\cot^{-1}x_0]^{-1}, \tag{53}$$

where

$$x_0 = \frac{b}{c}, \quad c = \frac{(a^2-b^2)^{\frac{1}{2}}}{b}. \tag{54}$$

When c is small and x_0 is large, (53) becomes

$$F = -6\pi(1+\tfrac{4}{5}\epsilon+\tfrac{2}{175}\epsilon^2+O(\epsilon^3)), \tag{55}$$

which is the small-λ limit of (52).

For large frequencies, $|\lambda| \gg 1$, equation (52) is dominated by the λ^2 term. This term, which gives the added-mass force on the spheroid, is given by

$$F = -\tfrac{2}{3}\pi\lambda^2(1+\tfrac{16}{5}\epsilon+\tfrac{604}{175}\epsilon^2+O(\epsilon^3))+O(\lambda). \tag{56}$$

The general formula for the added mass for an arbitrary oblate spheroid is given in Green (1833) by

$$F = -\tfrac{4}{3}\pi\frac{a^2}{b^2}\frac{(1+x_0^2)(1-x_0\cot^{-1}x_0)}{(1+x_0^2)x_0\cot x_0-x_0^2}. \tag{57}$$

Again, when c is small and x_0 is large, (57) reduces to (56).

Curiously, the $O(\lambda)$ corrections to (55) and (56) are the same, viz.

$$6\pi\lambda(1+\tfrac{8}{5}\epsilon+\tfrac{116}{175}\epsilon^2+O(\epsilon^3)). \tag{58}$$

This is identical in form to the Basset force on a sphere, and so should retain that name. However, if we carried the expansion further, the $O(\epsilon^3)$ term would be different at high and low frequencies, so for a general body the Basset force may need to be redefined.

There is an additional term in (52) which is $O(\lambda^2)$ for small λ and $O(1)$ for large λ so it only contributes to the second-order correction to (55) or (56). It is this term which prevents the force from being quadratic in λ and it is a mathematically different contribution to the hydrodynamic force. Although it is small, the term is conceptually

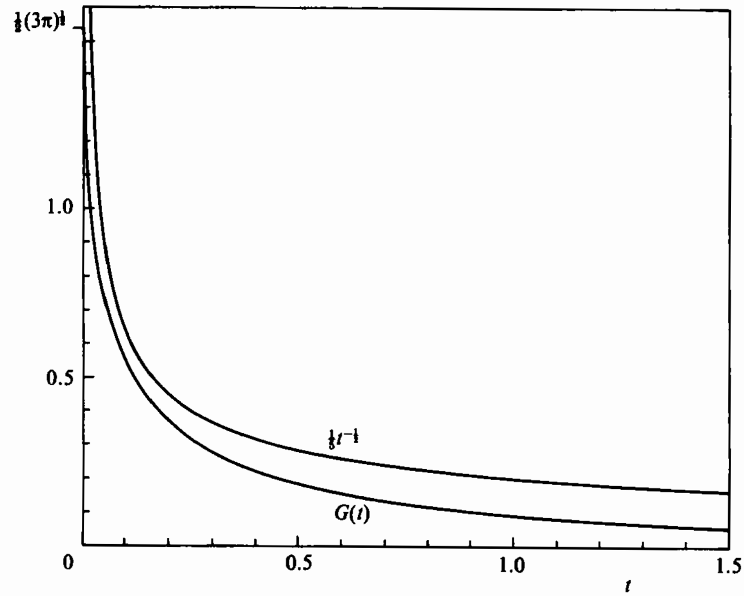

FIGURE 2. The memory functions $t^{-\frac{1}{2}}$ and $G(t)$ used in (59).

important because it indicates the presence of a complex interaction between body shape and frequency, which heretofore was not recognized. It is likely that this term will be more important for a general body whose shape departs significantly from a sphere, although its importance cannot be predicted from the present results.

5. Inverse Laplace transform

Equation (11) can be regarded as a Laplace transform of (5) with transform variable $s = \lambda^2$. If we allow the velocity of the body to have arbitrary time dependence, we can perform a superposition of the harmonic motions which is equivalent to an inverse Laplace transform. Suppose that $U(t)$ is zero for $t \leqslant 0$ and varies arbitrarily thereafter, then we can invert (52) formally to give the hydrodynamic force on the body, $F(t)$:

$$F(t) = -6\pi(1+\tfrac{4}{5}\epsilon+\tfrac{2}{175}\epsilon^2)\,U(t) - 6\pi^{\frac{1}{2}}(1+\tfrac{8}{5}\epsilon+\tfrac{116}{175}\epsilon^2)\int_0^t \frac{\mathrm{d}U}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{(t-\tau)^{\frac{1}{2}}}$$

$$-\tfrac{2}{3}\pi(1+\tfrac{16}{5}\epsilon+\tfrac{604}{175}\epsilon^2)\frac{\mathrm{d}U}{\mathrm{d}t} - \tfrac{8}{175}(\tfrac{1}{3}\pi)^{\frac{1}{2}}\epsilon^2\int_0^t \frac{\mathrm{d}U}{\mathrm{d}\tau}G(t-\tau)\,\mathrm{d}\tau + O(\epsilon^3) \quad (59)$$

with
$$G(t) = \mathrm{Im}\,\{(\pi\alpha)^{\frac{1}{2}}\,\mathrm{e}^{\alpha t}\,\mathrm{erfc}\,(\alpha t)^{\frac{1}{2}}\}, \quad (60)$$
and
$$\alpha = 3\mathrm{e}^{\frac{1}{3}\mathrm{i}\pi} = \tfrac{3}{2}(1+\surd 3\mathrm{i}). \quad (61)$$

The first three terms of (59) are the standard forms for the Stokes drag, Basset force and added-mass force respectively. The fourth term, like the Basset force, is a memory integral of the previous accelerations of the body, but the kernel function $G(t)$ is of a different nature to the $t^{-\frac{1}{2}}$ in the Basset force, as can be seen in figure 2. The two functions are asymptotically different for large t, and G is not infinite for small t. Thus the recent history is not emphasized in the second integral as it is in the Basset force, and in most cases the contribution from $G(t)$ will be very small. The behaviour of $G(t)$ is found from Abramowitz & Stegun (1965). For small t, we have a power series:

$$G(t) = (3\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\frac{1}{2}n+1)} (3t)^{\frac{1}{2}n} \sin (n+1)\tfrac{1}{6}\pi. \tag{62}$$

Thus, the limiting value is $G(0) = \frac{1}{2}(3\pi)^{\frac{1}{2}}$ and the initial slope is infinite. For large t, we have an asymptotic power series:

$$G(t) \sim t^{-\frac{1}{2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1.3\ldots(2n-1)}{(6t)^n} \sin \tfrac{1}{3}n\pi. \tag{63}$$

So $G(t)$ ultimately decays as $(1/(4\sqrt{3}))\,t^{-\frac{3}{2}}$.

This analysis has been for small ϵ, so we cannot conclude that similar results would apply for a general spheroid, or other body shape. In the general case, $G(t)$ can be any well-behaved function and the Basset force may not be so easily identifiable. The $O(\epsilon^3)$ term would give different behaviour for the $O(\lambda)$ terms at low and high frequency, so the behaviour of the memory function may be different. However, it will never be more singular for short time than the $t^{-\frac{1}{2}}$ in the Basset integral, whilst for long times, we might still expect $t^{-\frac{3}{2}}$ behaviour.

It is evident from the above that the sphere is a special case in which by coincidence the frequency dependence of the force decouples to some extent from the shape dependence. There does not appear to be a physical reason for this; rather it is a consequence of the mathematical simplicity of the spherical geometry. The dimensionless stream function for the oscillating sphere is (Stokes 1851)

$$\psi = -\tfrac{1}{2}\sin^2\theta \frac{1}{\lambda^2}\left[(\lambda^2+3\lambda+3)\frac{1}{r} - 3\left(\lambda^2+\frac{\lambda}{r}\right)\mathrm{e}^{-\lambda(r-1)}\right].$$

The coefficient of the $1/r$ term is determined by the boundary condition at $r = 1$. We see that the exponential term vanishes from ψ and all its derivatives at $r = 1$, so that the coefficients are determined in terms of powers of λ.

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